Hopf Algebras: A Basic Introduction

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Based upon the following textbooks:

Moss Eisenberg Sweedler,

Hopf algebras,

Mathematics Lecture Note Series, W. A. Benjamin, 1969

S. Dăscălescu, C. Năstăsescu, Ş. Raianu,

Hopf algebras: an introduction,

Monographs and Textbooks in Pure and Applied Mathematics 235,

Marcel Dekker, 2001

Tonny Albert Springer,

Linear algebraic groups,

Progress in Mathematics, Birkhäuser, 2nd Edition 1998

In this presentation,

$\mathbb{K}$ denotes a field, and

all tensor products are over $\mathbb{K}$, e.g., $V \otimes W = V \otimes_{\mathbb{K}} W$.

All rings and associative algebras are assumed to have identity.

Chapter 1.

Basic Definitions, Notions, and Examples

Definition of (associative) algebras over $\mathbb{K}$

There are many equivalent definitions for an (associative) algebra A over $\mathbb{K}$:

- A is a ring together with a ring homomorphism $\mathbb{K} \to A$ whose image is in the center of A.
- A is a $\mathbb{K}$-vector space together with a $\mathbb{K}$-bilinear operation $A \times A \to A$ such that $(xy)z = x(yz), \quad \forall x, y, z \in A$, in which A has multiplicative identity.

$$\vdots$$

What is a 'good' definition of algebras for us?

Among these equivalent ones we adopt the following (next page)

definition of algebras over $\mathbb{K}$ because it can be easily dualizable.

Definition of (associative) algebras over $\mathbb{K}$, continued

A is called an algebra over $\mathbb{K}$ if

A is a $\mathbb{K}$-vector space together with two $\mathbb{K}$-linear maps

$M : A \otimes A \to A$ and $u : \mathbb{K} \to A$ such that

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mathrm{Id} \otimes M"] \arrow[d, "M \otimes \mathrm{Id}"'] & A \otimes A \arrow[d, "M"] \\
A \otimes A \arrow[r, "M"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[dr, "M"] & \mathbb{K} \otimes A \arrow[l, "u \otimes \mathrm{Id}"'] \arrow[d, dash, "\simeq"] \\
A \otimes \mathbb{K} \arrow[u, "\mathrm{Id} \otimes u"] \arrow[r, dash, "\simeq"'] & A
\end{tikzcd}$$

commute, where $\mathrm{Id} : A \to A$ is the identity map.

We call M a *product* and u a *unit*, because

$xy := M(x \otimes y)$ and $1_A := u(1_{\mathbb{K}})$ play role as a usual

multiplication and identity in A.

Dualizing

By reversing all the directions of the arrows,

we obtain the notion of coalgebras over $\mathbb{K}$...

Definition of coalgebras (cogebras) over $\mathbb{K}$

A coalgebra C over $\mathbb{K}$ is a $\mathbb{K}$-vector space together with two $\mathbb{K}$-linear maps $\Delta : C \to C \otimes C$ and $\epsilon : C \to \mathbb{K}$ such that

$$
\begin{tikzcd}
C \otimes C \otimes C & & C \otimes C \arrow[ll, "\mathrm{Id} \otimes \Delta"'] \\
C \otimes C \arrow[u, "\Delta \otimes \mathrm{Id}"] & & C \arrow[ll, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C \otimes C \arrow[r, "\epsilon \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id} \otimes \epsilon"'] & \mathbb{K} \otimes C \arrow[d, dash, "\simeq"] \\
C \otimes \mathbb{K} \arrow[r, dash, "\simeq"'] & C \arrow[ul, "\Delta"']
\end{tikzcd}
$$

commute.

We call Δ a *coproduct* and ϵ a *counit*.

The identity $(\mathrm{Id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{Id}) \circ \Delta$ from the first diagram is referred to as the "*coassociativity*".

Commutativity and Cocommutativity

- An algebra (A, M, u) is said to be *commutative* if

$$\begin{tikzcd}
A \otimes A \arrow[rr, "x \otimes y \mapsto y \otimes x"] \arrow[dr, "M"'] & & A \otimes A \arrow[dl, "M"] \\
& A &
\end{tikzcd}$$

  commutes.

- A coalgebra (C, Δ, ϵ) is said to be *cocommutative* if

$$\begin{tikzcd}
C \otimes C \arrow[rr, "x \otimes y \mapsto y \otimes x"] & & C \otimes C \\
& C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

  commutes.

Examples of coalgebras (I)

Ex. 1. 'Group-like coalgebra'

Let S be a set and V a $\mathbb{K}$-space with the set S as basis.

Define $\Delta : V \to V \otimes V$ and $\epsilon : V \to \mathbb{K}$ by

$\Delta(s) := s \otimes s$ and $\epsilon(s) := 1$, $\forall s \in S$.

Then V becomes a (cocomutative) coalgebra over $\mathbb{K}$.

Ex. 2. 'Devided power coalgebra'

Let D be a $\mathbb{K}$-vector space with a basis $\{d_m | m = 0, 1, 2, \cdots\}$.

Define $\Delta : D \to D \otimes D$ and $\epsilon : D \to \mathbb{K}$ by

$$\Delta(d_m) := \sum_{k=0}^{m} d_k \otimes d_{m-k} \text{ and } \epsilon(d_m) := \delta_{0,m} \ , \quad \forall m = 0, 1, 2, \cdots .$$

Then D becomes a (cocomutative) coalgebra.

Examples of coalgebras (II)

Ex. 3. 'Matrix coalgebra'

Let $\{e_{ij}\}_{1\leq i,j\leq n}$ be the canonical basis for $M := \mathrm{Mat}_n(\mathbb{K})$.

Then M is a coalgebra if $\Delta : M \to M \otimes M$ and $\epsilon : M \to \mathbb{K}$ are

$$\Delta(e_{ij}) := \sum_{k=1}^{n} e_{ik} \otimes e_{kj} \quad \text{and} \quad \epsilon(e_{ij}) := \delta_{ij}.$$

Ex. 4. 'Incidence coalgebra'

Let $(P, \leq)$ be a locally finite partially ordered set, i.e,

for any $x, y \in P$ with $x \leq y$, the set $\{z|x \leq z \leq y\}$ is finite.

If V is a $\mathbb{K}$-vector space with $\{(x,y) \in P \times P | x \leq y\}$ as basis,

$$\Delta((x,y)) := \sum_{x\leq z\leq y} (x,z) \otimes (z,y), \text{ and } \epsilon((x,y)) := \delta_{x,y},$$

then V becomes a coalgebra.

Morphisms of algebras and coalgebras

- A $\mathbb{K}$-linear map $f : A \to B$ of algebras is a *morphism* if

$$\begin{tikzcd} A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "M_A"'] & B \otimes B \arrow[d, "M_B"] \\ A \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "f"] & & B \\ & \mathbb{K} \arrow[ul, "u_A"] \arrow[ur, "u_B"'] & \end{tikzcd}$$

commute.

- A $\mathbb{K}$-linear map $g : C \to D$ of coalgebras is a *morphism* if

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "g \otimes g"] & D \otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "g"] \arrow[dr, "\epsilon_C"'] & & D \arrow[dl, "\epsilon_D"] \\ & \mathbb{K} & \end{tikzcd}$$

commute.

Generalized associativity

- In algebra A, we know the "generalized associativity", e.g.,

$$(ab)((cd)((ef)g)) = a(b(((cd)e)(fg))) \quad \forall a, b, c, d, e, f, g \in A.$$

Systemically, we can think of it as following.

In algebra (A, M, u), put $M^1 := M$ and define recursively

$$M^n : \underbrace{A \otimes \cdots \otimes A}_{n+1 \text{ times}} \to A \text{ by } M^n := M^{n-1} \circ (M \otimes \underbrace{\mathrm{Id} \otimes \cdots \otimes \mathrm{Id}}_{n-1 \text{ times}}).$$

Then we have "*generalized associativity*":

For any $n \geq 2$, $k \in \{1, \cdots, n-1\}$, and $p \in \{0, \cdots, n-k\}$,

$$M^n = M^{n-k} \circ (\underbrace{\mathrm{Id} \otimes \cdots \otimes \mathrm{Id}}_{p \text{ times}} \otimes M^k \otimes \underbrace{\mathrm{Id} \otimes \cdots \otimes \mathrm{Id}}_{n-k-p \text{ times}}) \text{ holds.}$$

Generalized coassociativity

- In coalgebra (C, Δ, ϵ), put $\Delta^1 := \Delta$ and define recursively
  $\Delta^n : C \to \underbrace{C \otimes \cdots \otimes C}_{n+1 \text{ times}}$ by $\Delta^n := (\Delta \otimes \underbrace{\mathrm{Id} \otimes \cdots \otimes \mathrm{Id}}_{n-1 \text{ times}}) \circ \Delta^{n-1}$.

  Then we have "*generalized coassociativity*":

  For any $n \geq 2$, $k \in \{1, \cdots, n-1\}$, and $p \in \{0, \cdots, n-k\}$,

  $\Delta^n = (\underbrace{\mathrm{Id} \otimes \cdots \otimes \mathrm{Id}}_{p \text{ times}} \otimes \Delta^k \otimes \underbrace{\mathrm{Id} \otimes \cdots \otimes \mathrm{Id}}_{n-k-p \text{ times}}) \circ \Delta^{n-k}$ holds.

Product vs Coproduct

- We can view a product map as "law of composition", i.e.,

$$z := xy = M(x \otimes y).$$

  The resulting quantity $z = xy$ is more simple than x and y in the sense that the number of quantities decreases.

- However, a coproduct map is a "law of decomposition", i.e.,

$$\Delta(x) = \sum_{i,j} x_{1i} \otimes x_{2j}.$$

  Usually, Δ produces lots of resulting quantities x_{1i} and x_{2j}, and hence we need many summation indicies for them.

The sigma notation (a.k.a. Sweedler notation)

"*WARNING!!* The notation introduced in this section plays a key role in the sequel..."

– M. E. Sweedler in his book 'Hopf algebras', Section 1.2.

For coproduct Δ or generalized coproduct Δ^n, the sigma notation just *suppresses summation indicies* of resulting quantities.

For instance, if

$$\Delta(x) = \sum_{i,j} x_{1i} \otimes x_{2j} \quad \text{and} \quad \Delta^3(x) = \sum_{i,j,k,\ell} x_{1i} \otimes x_{2j} \otimes x_{3k} \otimes x_{4\ell},$$

then the sigma notation suggests to write the above equations as

$$\Delta(x) = \sum x_1 \otimes x_2 \quad \text{and} \quad \Delta^3(x) = \sum x_1 \otimes x_2 \otimes x_3 \otimes x_4.$$

Examples for use of the sigma notation

Let (C, Δ, ϵ) be a coalgebra and $x \in C$.

Ex. 1. The coassociativity $(\mathrm{Id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{Id}) \circ \Delta = \Delta^2$ is

$$\sum x_1 \otimes (x_2)_1 \otimes (x_2)_2 = \sum (x_1)_1 \otimes (x_1)_2 \otimes x_2 = \sum x_1 \otimes x_2 \otimes x_3.$$

Ex. 2. The defining identity of the counit ϵ is

$$\sum \epsilon(x_1) \otimes x_2 = x = \sum x_1 \otimes \epsilon(x_2).$$

Ex. 3. A $\mathbb{K}$-linear map $g : C \to D$ is a coalgebra morphism iff

$$\sum g(x_1) \otimes g(x_2) = \sum g(x)_1 \otimes g(x)_2 \quad \text{and} \quad \epsilon_C(x) = \epsilon_D(g(x)).$$

Warm up practice

If (C, Δ, ϵ) be a coalgebra, can you verify the following identities?

Exer. 1. $\sum \epsilon(x_2) \otimes \Delta(x_1) = \Delta(x)$.

Exer. 2. $\sum \Delta(x_2) \otimes \epsilon(x_1) = \Delta(x)$.

Exer. 3. $\sum x_1 \otimes \epsilon(x_3) \otimes x_2 = \Delta(x)$.

Exer. 4. $\sum x_1 \otimes x_3 \otimes \epsilon(x_2) = \Delta(x)$.

Exer. 5. $\sum \epsilon(x_1) \otimes x_3 \otimes x_2 = \sum x_2 \otimes x_1$.

Exer. 6. $\sum \epsilon(x_1) \otimes \epsilon(x_3) \otimes x_2 = x$.

Computation rule using the sigma notation

(C, Δ, ϵ) : a coalgebra over $\mathbb{K}$

$f : \underbrace{C \otimes \cdots \otimes C}_{n+1 \text{ times}} \to C$: a $\mathbb{K}$-linear map

$\overline{f} : C \to C$: the composition map $C \xrightarrow{\Delta^n} \underbrace{C \otimes \cdots \otimes C}_{n+1 \text{ times}} \xrightarrow{f} C$.

$g : \underbrace{C \otimes \cdots \otimes C}_{k+1 \text{ times}} \to C$: a $\mathbb{K}$-linear map with $k \geq n$

$\Longrightarrow$ The following general "*computation rule*" holds:

For any $x \in C$ and $1 \leq j \leq n+1$

$$\sum g(x_1 \otimes \cdots \otimes x_{j-1} \otimes f(x_j \otimes \cdots \otimes x_{j+n}) \otimes x_{j+n+1} \otimes \cdots \otimes x_{k+n+1})$$

$$= \sum g(x_1 \otimes \cdots \otimes x_{j-1} \otimes \overline{f}(x_j) \otimes x_{j+1} \otimes \cdots \otimes x_{k+1}).$$

Proof of computation rule

Proof.

$$\sum g(x_1 \otimes \cdots \otimes x_{j-1} \otimes f(x_j \otimes \cdots \otimes x_{j+n}) \otimes x_{j+n+1} \otimes \cdots \otimes x_{k+n+1})$$

$$= g \circ (\mathrm{Id}^{\otimes j-1} \otimes f \otimes \mathrm{Id}^{\otimes k-j+1}) \circ \Delta^{k+n}(x)$$

$$= g \circ (\mathrm{Id}^{\otimes j-1} \otimes f \otimes \mathrm{Id}^{\otimes k-j+1}) \circ (\mathrm{Id}^{\otimes j-1} \otimes \Delta^{n} \otimes \mathrm{Id}^{\otimes k-j+1}) \circ \Delta^{k}(x)$$

$$= g \circ (\mathrm{Id}^{\otimes j-1} \otimes (f \circ \Delta^{n}) \otimes \mathrm{Id}^{\otimes k-j+1}) \circ \Delta^{k}(x)$$

$$= g \circ (\mathrm{Id}^{\otimes j-1} \otimes \overline{f} \otimes \mathrm{Id}^{\otimes k-j+1}) \circ \Delta^{k}(x)$$

$$= \sum g(x_1 \otimes \cdots \otimes x_{j-1} \otimes \overline{f}(x_j) \otimes x_{j+1} \otimes \cdots \otimes x_{k+1}).$$

□

Chapter 2.

Duality between Algebras and Coalgebras

Review: Some linear algebra (I)

V, $V^* := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$: a $\mathbb{K}$-vector space & its dual space

$\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{K}$: the natural pairing, i.e., $\langle f, v \rangle := f(v)$

If $A \subseteq V$ then $A^\perp := \{f \in V^* \mid \langle f, v \rangle = 0, \forall v \in A\}$.

If $B \subseteq V^*$ then $B^\perp := \{v \in V \mid \langle f, v \rangle = 0, \forall f \in B\}$.

$\Longrightarrow$ $V^\perp = 0$ and $V^{*\perp} = 0$.

$\Longrightarrow$ If $\varphi : V \to W$ is a $\mathbb{K}$-linear map of $\mathbb{K}$-vector spaces,

then its *transpose* $\varphi^* : W^* \to V^*$ is uniquely defined by

$$\langle \varphi^*(g), v \rangle = \langle g, \varphi(v) \rangle \quad \text{for all } g \in W^* \text{ and } v \in V.$$

(Note that it is just $\varphi^* : W^* \to V^*$, $g \mapsto g \circ \varphi$.)

Review: Some linear algebra (II)

We define $\rho : V^* \otimes W^* \to (V \otimes W)^*$ by

$$\langle \rho(f \otimes g), v \otimes w \rangle := \langle f, v \rangle \langle g, w \rangle, \quad \forall f \in V^*, g \in W^*, v \in V, w \in W,$$

namely, $\rho(f \otimes g)(v \otimes w) := f(v)g(w)$.

$\Longrightarrow$ Recall that the map ρ is a canonical injection.

Moreover if one of V and W is finite dimensional,

then the map ρ becomes a $\mathbb{K}$-linear isomorphism.

The dual algebra of a coalgebra

Let (C, Δ, ϵ) be a coalgebra over $\mathbb{K}$ and

$C^* = \mathrm{Hom}_{\mathbb{K}}(C, \mathbb{K})$ be its dual space.

We can define $M : C^* \otimes C^* \to C^*$ and $u : \mathbb{K} \to C^*$ by

$M : C^* \otimes C^* \xrightarrow{\rho} (C \otimes C)^* \xrightarrow{\Delta^*} C^*$ and

$u : \mathbb{K} \xrightarrow{\simeq} \mathbb{K}^* \xrightarrow{\epsilon^*} C^*$.

Proposition

1. (C^*, M, u) is an algebra over $\mathbb{K}$.
2. If $g : C \to D$ is a morphism of coalgebras then $g^* : D^* \to C^*$ is a morphism of algebras.

The dual coalgebra of a finite dimensional algebra

Let (A, M, u) be a finite dimensional algebra over $\mathbb{K}$ and

$A^* = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$ be its dual space.

In this case, the map $\rho : A^* \otimes A^* \to (A \otimes A)^*$ is bijective.

Thus we can define $\Delta : A^* \to A^* \otimes A^*$ and $\epsilon : A^* \to \mathbb{K}$ by

$\Delta : A^* \xrightarrow{M^*} (A \otimes A)^* \xrightarrow{\rho^{-1}} A^* \otimes A^*$ and

$\epsilon : A^* \xrightarrow{u^*} \mathbb{K}^* \xrightarrow{\simeq} \mathbb{K}$.

Proposition

1. (A^*, Δ, ϵ) is a coalgebra over $\mathbb{K}$.
2. If $f : A \to B$ is a morphism of algebras then

   $f^* : B^* \to A^*$ is a morphism of coalgebras.

Categorical duality for finite dimensional case

(A, M, u) : a finite dimensional algebra

(C, Δ, ϵ) : a finite dimensional coalgebra

If V is a finite dimensional vector space, then recall that

$\mathcal{E} : V \to V^{**}$, $\mathcal{E}(v)(f) := f(v)$, $\forall v \in V, f \in V^*$ is an isomorphism.

Proposition

1. $\mathcal{E} : A \to A^{**}$ is an isomorphism of algebras;
2. $\mathcal{E} : C \to C^{**}$ is an isomorphism of coalgebras.

$\Longrightarrow$ The category **F Coalg** is anti-equivalent to the category **F Alg**.

Also, we have **F Cocomm.Coalg** $\underset{\text{anti}}{\overset{\simeq}{\longleftrightarrow}}$ **F Comm.Alg**.

Sub-coalgebras of a coalgebra & its duality

Let (C, Δ, ϵ) be a coalgebra.

If V is a subspace of C that satisfies $\Delta(V) \subseteq V \otimes V$, then clearly $(V, \Delta|_V, \epsilon|_V)$ becomes a coalgebra and it is easy to check that the inclusion map $V \hookrightarrow C$ is a morphism of coalgebras.

This fact naturally leads to the following definition:

Definition

A subspace $V \subseteq C$ is called a *sub-coalgebra* if $\Delta(V) \subseteq V \otimes V$.

Proposition

1. If $V \subseteq C$ is a sub-coalgebra, $V^\perp$ is a (two-sided) ideal of C^*.
2. If $J \subseteq C^*$ is a (two-sided) ideal, $J^\perp$ is a sub-coalgebra of C.

Coideals of a coalgebra & its duality

Let (C, Δ, ϵ) be a coalgebra.

Definition

A subspace $V \subseteq C$ is called a (two-sided) *coideal* if

1. $\Delta(V) \subseteq V \otimes C + C \otimes V$;
2. $\epsilon(V) = 0$.

Proposition

1. If $V \subseteq C$ is a coideal, $V^{\perp}$ is a subalgebra of C^*.
2. If $B \subseteq C^*$ is a subalgebra, $B^{\perp}$ is a coideal of C.

Kernel and image for a morphism of coalgebras

Let $g : C \to D$ be a morphism of coalgebras.

Proposition

1. $\operatorname{Ker} g$ is a coideal in C;
2. $\operatorname{Im} g$ is a sub-coalgebra in D.

If $J \subseteq C$ is a coideal, there is a unique coalgebra structure on C/J such that $\pi : C \to C/J$ is a morphism of coalgebras.

Homomorphism Theorem

If $J \subseteq \operatorname{Ker} g$ is a coideal, there is a unique morphism of coalgebras $\overline{g} : C/J \to D$ such that $\overline{g} \circ \pi = g$. In particular, $C/\operatorname{Ker} g \cong \operatorname{Im} g$.

Left and right coideals of a coalgebra & its duality

Let (C, Δ, ϵ) be a coalgebra.

Definition

1. A subspace $V \subseteq C$ is called a *left coideal* if $\Delta(V) \subseteq C \otimes V$;
2. A subspace $V \subseteq C$ is called a *right coideal* if $\Delta(V) \subseteq V \otimes C$.

Proposition

1. If $V \subseteq C$ is a left (right) coideal, $V^{\perp}$ is a left (right) ideal in C^*;
2. If $J \subseteq C^*$ is a left (right) ideal, $J^{\perp}$ is a left (right) coideal in C.

Caution!!

A coideal need not be either a left or a right coideal.

Furthermore, if $V \subseteq C$ is both a left and right coideal,

then V is a sub-coalgebra and not a coideal unless $V = 0$.

This is because $(V \otimes C) \cap (C \otimes V) = V \otimes V$.

(Or, simply, by duality.)

Chapter 3.

Bialgebras and Hopf Algebras

Review: The tensor product of two algebras is an algebra.

(A, M_A, u_A), (B, M_B, u_B) : algebras over $\mathbb{K}$

$T : A \otimes B \to B \otimes A$: the 'twist' map, i.e., $a \otimes b \mapsto b \otimes a$

We can define $M_{A\otimes B}$ by

$$M_{A\otimes B} : A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{Id} \otimes T \otimes \mathrm{Id}} A \otimes A \otimes B \otimes B \xrightarrow{M_A \otimes M_B} A \otimes B.$$

Also, we can define $u_{A\otimes B}$ by

$$u_{A\otimes B} : \mathbb{K} \xrightarrow{\simeq} \mathbb{K} \otimes \mathbb{K} \xrightarrow{u_A \otimes u_B} A \otimes B.$$

Proposition

$(A \otimes B, M_{A\otimes B}, u_{A\otimes B})$ is an algebra.

The tensor product of two coalgebras is a coalgebra.

$(C, \Delta_C, \epsilon_C)$, $(D, \Delta_D, \epsilon_D)$: coalgebras over $\mathbb{K}$

$T : C \otimes D \to D \otimes C$: the 'twist' map, i.e., $c \otimes d \mapsto d \otimes c$

We can define $\Delta_{C \otimes D}$ by

$$\Delta_{C \otimes D} : C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{\mathrm{Id} \otimes T \otimes \mathrm{Id}} C \otimes D \otimes C \otimes D.$$

Also, we can define $\epsilon_{C \otimes D}$ by

$$\epsilon_{C \otimes D} : C \otimes D \xrightarrow{\epsilon_C \otimes \epsilon_D} \mathbb{K} \otimes \mathbb{K} \xrightarrow{\simeq} \mathbb{K}.$$

Proposition

$(C \otimes D, \Delta_{C \otimes D}, \epsilon_{C \otimes D})$ is a coalgebra.

Definition of bialgebras (bigebras)

Suppose there is a system $(H, M, u, \Delta, \epsilon)$ such that

(H, M, u) is an algebra and (H, Δ, ϵ) is a coalgebra.

Proposition

The following are equivalent:

(A) $M : H \otimes H \to H$ and $u : \mathbb{K} \to H$ are coalgebra morphisms;

(B) $\Delta : H \to H \otimes H$ and $\epsilon : H \to \mathbb{K}$ are algebra morphisms.

Proof. See the diagrams in next page. $\square$

Definition

$(H, M, u, \Delta, \epsilon)$ is called a *bialgebra* if one of (A) and (B) holds.

$$
\begin{tikzcd}
 & H \arrow[r, "\Delta"] & H \otimes H \\
H \otimes H \arrow[ur, "M"] \arrow[dr, "\Delta \otimes \Delta"'] & & H \otimes H \otimes H \otimes H \arrow[u, "M \otimes M"'] \\
 & H \otimes H \otimes H \otimes H \arrow[ur, "\mathrm{Id} \otimes T \otimes \mathrm{Id}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
H \otimes H \arrow[r, "\epsilon \otimes \epsilon"] \arrow[d, "M"'] & \mathbb{K} \otimes \mathbb{K} \arrow[d, "\simeq"', "M_{\mathbb{K}}"] \\
H \arrow[r, "\epsilon"'] & \mathbb{K}
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[d, "\Delta"'] & \mathbb{K} \arrow[l, "u"'] \arrow[d, "\simeq"', "\Delta_{\mathbb{K}}"] \\
H \otimes H & \mathbb{K} \otimes \mathbb{K} \arrow[l, "u \otimes u"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{K} \arrow[rr, "\mathrm{Id}"] \arrow[dr, "u"'] & & \mathbb{K} \\
 & H \arrow[ur, "\epsilon"'] &
\end{tikzcd}
$$

Convolution algebra: $\mathrm{Hom}_{\mathbb{K}}(C, A)$

(A, M, u) : an algebra over $\mathbb{K}$

(C, Δ, ϵ) : a coalgebra over $\mathbb{K}$

$H := \mathrm{Hom}_{\mathbb{K}}(C, A)$: the set of all $\mathbb{K}$-linear maps from C to A

We define so called the '*convolution product*' $* : H \otimes H \to H$ by

$$* : H \otimes H \hookrightarrow \mathrm{Hom}_{\mathbb{K}}(C \otimes C, A \otimes A) \xrightarrow{\mathrm{Hom}(\Delta, M)} H,$$

where the first map is a canonical injection, and

the second map $\mathrm{Hom}(\Delta, M)$ is the composition map defined by

$$\mathrm{Hom}(\Delta, M) : \varphi \mapsto M \circ \varphi \circ \Delta.$$

Identity element in the convolution algebra $\mathrm{Hom}_{\mathbb{K}}(C, A)$

Similarly, $\epsilon : C \to \mathbb{K}$ and $u : \mathbb{K} \to A$ induce $\eta : \mathbb{K} \to H$ defined by

$$\eta : \mathbb{K} \cong \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}, \mathbb{K}) \xrightarrow{\mathrm{Hom}(\epsilon, u)} H = \mathrm{Hom}_{\mathbb{K}}(C, A),$$

where $\mathrm{Hom}(\epsilon, u) : \varphi \mapsto u \circ \varphi \circ \epsilon$.

Consequently, we obtain the following result:

Proposition

1. $(\mathrm{Hom}_{\mathbb{K}}(C, A), *, \eta)$ is an algebra over $\mathbb{K}$;
2. The identity element in $\mathrm{Hom}_{\mathbb{K}}(C, A)$ is $\eta(1_{\mathbb{K}}) = u \circ \epsilon$.

Definition of Hopf algebras

$(H, M, u, \Delta, \epsilon)$: a bialgebra over $\mathbb{K}$

Put $H^A := (H, M, u)$ and $H^C := (H, \Delta, \epsilon)$.

Definition

$(H, M, u, \Delta, \epsilon)$ is a *Hopf algebra* if $\mathrm{Id} : H \to H$ has inverse $S : H \to H$ in the algebra $(\mathrm{Hom}_{\mathbb{K}}(H^C, H^A), *, \eta)$. S is called the *antipode*.

In other words, there is $S : H \to H$ commuting the following diagram:

$$\begin{tikzcd}
 & H \otimes H \arrow[rr, "S \otimes \mathrm{Id}"] & & H \otimes H \arrow[dr, "M"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & \mathbb{K} \arrow[rr, "u"] & & H \\
 & H \otimes H \arrow[rr, "\mathrm{Id} \otimes S"'] & & H \otimes H \arrow[ur, "M"'] &
\end{tikzcd}$$

Examples of Hopf algebras (I)

Ex. 1. Group algebra

Let G be a group and $\mathbb{K}G$ be a group algebra over $\mathbb{K}$.

$\mathbb{K}G$ is a bialgebra if we endow $\mathbb{K}G$ with 'group-like coalgebra'.

$\mathbb{K}G$ is a Hopf algebra with $S : \mathbb{K}G \to \mathbb{K}G$, $g \mapsto g^{-1}$, $\forall g \in G$.

It is cocommutative, and it is commutative iff G is abelian.

Ex. 2. The set $\mathbb{K}^G$ of all functions from a finite group G to $\mathbb{K}$

$\mathbb{K}^G$ is an algebra with pointwise addition and multiplication

and a coalgebra with $\Delta(\varphi)(g, h) := \varphi(gh)$ and $\epsilon(\varphi) := \varphi(1_G)$.

$\mathbb{K}^G$ is a Hopf algebra with $S(\varphi)(g) := \varphi(g^{-1})$.

It is commutative, and it is cocomutative iff G is abelian.

Examples of Hopf algebras (II)

Ex. 3. Tensor algebra & its families

Let $\mathrm{T}(V) = \bigoplus_{j=0}^{\infty} V^{\otimes j}$ be a tensor algebra over a $\mathbb{K}$-space V.

If, for all $v \in V$, we define $\Delta(v) := 1 \otimes v + v \otimes 1$, $\epsilon(v) := 0$,

and $S(v) := -v$, then $\mathrm{T}(V)$ is a cocomutative Hopf algebra.

Symmetric algebra and Exterior algebra are Hopf algebras.

Ex. 4. Universal enveloping algebra of a Lie algebra

Let $\mathrm{U}(\mathfrak{g})$ be a U.E.A. of a Lie algebra $\mathfrak{g}$ over $\mathbb{K}$.

If, for all $X \in \mathfrak{g}$, we define $\Delta(X) := 1 \otimes X + X \otimes 1$, $\epsilon(X) := 0$,

and $S(X) := -X$, then $\mathrm{U}(\mathfrak{g})$ is a cocomutative Hopf algebra.

It is commutative if and only if $\mathfrak{g}$ is abelian.

Examples of Hopf algebras (III)

Ex. 5. SWEEDLER's 4-dimensional Hopf algebra

Assume that $\operatorname{char} \mathbb{K} \neq 2$.

If H is generated as an algebra by c and x by the relations

$$c^2 = 1, \quad x^2 = 0, \quad xc = -cx,$$

then H is a 4-dimensional $\mathbb{K}$-space with basis $\{1, c, x, cx\}$.

The coalgebra structure of H is defined by

$$\Delta(c) := c \otimes c,\ \Delta(x) := c \otimes x + x \otimes 1,\ \epsilon(c) := 1,\ \epsilon(x) := 0.$$

If $S(c) := c^{-1}$, $S(x) := -cx$, then H is a Hopf algebra.

This is the smallest example which is both non-commutative and non-cocommutative.

Chapter 4.

Duality between

Linear Algebraic Groups and Hopf Algebras

From now on,

we suppose that $\mathbb{K}$ is algebraically closed.

Linear algebraic groups (=Affine algebraic groups)

Definition

An *algebraic group* G is an algebraic variety (over $\mathbb{K}$) which is also a group such that the maps defining the group structure $\mu : G \times G \to G, (g, h) \mapsto gh$ and $\iota : G \to G, g \mapsto g^{-1}$ are morphisms of varieties. (Here, $G \times G$ is the product of varieties.)

Definition

An algebraic group is called *linear* if the underlying variety is affine.

Definition

A *homomorphism* $G \to G'$ of algebraic groups is defined as a variety morphism which is also a group homomorphism.

Review: HILBERT's Nullstellensatz

In algebraic geometry, there is a well-known (anti-)correspondence between algebra and geometry via Nullstellensatz.

Geometry	$\leftrightarrow$	Algebra
Affine variety V	$\leftrightarrow$	Affine algebra $\mathbb{K}[V]$
Points in V	$\leftrightarrow$	Maximal ideals in $\mathbb{K}[V]$
Irr. closed sub-varieties of V	$\leftrightarrow$	Prime ideals in $\mathbb{K}[V]$
Variety morphism $V_1 \to V_2$	$\leftrightarrow$	Algebra morphism $\mathbb{K}[V_2] \to \mathbb{K}[V_1]$
(Categorical) Product $V_1 \times V_2$	$\leftrightarrow$	Coproduct $\mathbb{K}[V_1] \otimes \mathbb{K}[V_2]$
Combinatorial dimension	$\leftrightarrow$	Krull Dimension
$\vdots$	$\vdots$	$\vdots$

Duality between linear algebraic groups & Hopf algebras

Linear algebraic groups G	$\leftrightarrow$	(comm.) Hopf algebra $\mathbb{K}[G]$
Affine variety G	$\leftrightarrow$	Affine algebra $\mathbb{K}[G]$
Map $G_1 \to G_2$	$\leftrightarrow$	Map $\mathbb{K}[G_2] \to \mathbb{K}[G_1]$
(Categorical) Product $G \times G$	$\leftrightarrow$	Coproduct $\mathbb{K}[G] \otimes \mathbb{K}[G]$
$\mu : G \times G \to G$	$\leftrightarrow$	$\mu^0 = \Delta : \mathbb{K}[G] \to \mathbb{K}[G] \otimes \mathbb{K}[G]$
$\iota : G \to G$	$\leftrightarrow$	$\iota^0 = S : \mathbb{K}[G] \to \mathbb{K}[G]$
Associativity of μ	$\xleftrightarrow{Ax.1}$	Coassociativity of Δ
Existence of identity	$\xleftrightarrow{Ax.2}$	Counitary property
Existence of inverse	$\xleftrightarrow{Ax.3}$	Antipodal property

For $(\mathbb{K}[G], M, u, \Delta, \epsilon, S)$, $M(\varphi, \psi)(g) = \varphi(g)\psi(g)$, $u(1_{\mathbb{K}}) = 1_{\mathbb{K}}$, $\Delta(\varphi)(g, h) = \varphi(gh)$, $\epsilon(\varphi) = \varphi(1_G)$, and $S(\varphi)(g) = \varphi(g^{-1})$.

Put $A := \mathbb{K}[G]$ and $M^0 = \mathrm{diag} : G \to G,\ g \mapsto (g, g)$.

$$\begin{tikzcd}
G \times G \times G \arrow[r, "\mathrm{Id} \times \mu"] \arrow[d, "\mu \times \mathrm{Id}"'] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"] & G
\end{tikzcd}
\quad \overset{Ax.1}{\longleftrightarrow} \quad
\begin{tikzcd}
A \otimes A \otimes A & A \otimes A \arrow[l, "\mathrm{Id} \otimes \Delta"'] \\
A \otimes A \arrow[u, "\Delta \otimes \mathrm{Id}"] & A \arrow[l, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"'] & G \arrow[d, "g \mapsto (1_G, g)"] \\
G \arrow[r, "g \mapsto (g, 1_G)"'] & G \times G \arrow[ul, "\mu"']
\end{tikzcd}
\quad \overset{Ax.2}{\longleftrightarrow} \quad
\begin{tikzcd}
A \arrow[dr, "\Delta"] & \mathbb{K} \otimes A \arrow[l, "\simeq"'] \\
A \otimes \mathbb{K} \arrow[u, "\simeq"] & A \otimes A \arrow[l, "\mathrm{Id} \otimes \epsilon"] \arrow[u, "\epsilon \otimes \mathrm{Id}"']
\end{tikzcd}$$

$$\begin{tikzcd}
& G \times G \arrow[rr, "\iota \times \mathrm{Id}"] & & G \times G \arrow[dr, "\mu"] & \\
G \arrow[ur, "\mathrm{diag}"] \arrow[dr, "\mathrm{diag}"'] \arrow[rr, "g \mapsto 1_{\mathbb{K}}"] & & \mathbb{K} \arrow[rr, "1_{\mathbb{K}} \mapsto 1_G"] & & G \\
& G \times G \arrow[rr, "\mathrm{Id} \times \iota"'] & & G \times G \arrow[ur, "\mu"'] &
\end{tikzcd}
\quad \overset{Ax.3}{\longleftrightarrow} \quad
\begin{tikzcd}
& A \otimes A \arrow[dl, "M"'] & & A \otimes A \arrow[ll, "S \otimes \mathrm{Id}"'] & \\
A & & \mathbb{K} \arrow[ll, "u"'] & & A \arrow[ll, "\epsilon"'] \arrow[ul, "\Delta"'] \arrow[dl, "\Delta"] \\
& A \otimes A \arrow[ul, "M"] & & A \otimes A \arrow[ll, "\mathrm{Id} \otimes S"] &
\end{tikzcd}$$

Final comment:

The study of 'Quantum groups' (they are some kind of Hopf algebras) is a study for deformation of this duality between linear algebraic groups and Hopf algebras.

Thank you for your attention!

Enjoy Hopf algebra theory!!