How can we define the fractional radial derivative ?

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Outline

Definition

Let $\mathbb{C}^n$ be the complex n-space and $dxdy$ be the ordinary volume measure on $\mathbb{C}^n$. Let $d\mu$ be a Gaussian measure defined by

$$d\mu(z) = \frac{1}{\pi^n} e^{-|z|^2} dxdy.$$

The Fock space, denoted by $F^2 := F^2(\mathbb{C}^n)$, is then the space $L^2(\mathbb{C}^n) \cap H(\mathbb{C}^n)$. Being considered as a closed subspace of $L^2(\mathbb{C}^n)$, the space $F^2(\mathbb{C}^n)$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ given by

$$\langle f, g \rangle := \int_{\mathbb{C}^n} f(z)\overline{g(z)}\, d\mu(z)$$

and

$$\|f\|^2 := \int_{\mathbb{C}^n} |f(z)|^2\, d\mu(z)$$

for $f, g \in F^2$.

Definition

Here, we are using the standard multi-index notation. Namely, given an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers, $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ where ∂_j denotes the partial differentiation with respect to the j-th component.
We define

$$e_\alpha(z) = \frac{z^\alpha}{\|z^\alpha\|_{F^2}}$$

Then $\{e_\alpha : \alpha \in \mathbb{N}_0^n\}$ is an orthonormal basis for F^2.

For $f \in F^2$, let

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha e_\alpha(z)$$

be the orthonormal decomposition of f.

Definition of the radial derivative

We define the radial derivative $\mathcal{R}f$ by

$$\mathcal{R}f := \sum_{j=1}^{n} \left(2z_j \frac{\partial}{\partial z_j} f + 1 \right) \text{ for } f \in H(\mathbb{C}^n).$$

Then $\mathcal{R}$ is a self-adjoint operator.

Definition of the spectrum

$$\sigma(\mathcal{R}) = \left\{ \lambda \in \mathbb{C} \mid \nexists (\lambda I - \mathcal{R})^{-1} : \text{bounded} \right\}$$
$$= \sigma_p(\mathcal{R}) \sqcup \sigma_c(\mathcal{R}) \sqcup \sigma_r(\mathcal{R})$$

where

(Point Spectrum)

$$\sigma_p(\mathcal{R}) := \{\lambda \in \sigma(\mathcal{R}) \subseteq \mathbb{C} : \ker(\mathcal{R} - \lambda I) \neq \{0\}\}$$

(Continuous Spectrum)

$$\sigma_c(\mathcal{R}) := \left\{ \lambda \in \sigma(\mathcal{R}) \subseteq \mathbb{C} : \ker(\mathcal{R} - \lambda I) = \{0\} \textit{ and } \overline{\operatorname{Range}(\mathcal{R} - \lambda I)} = F^2 \right\}$$

(Residual Spectrum)

$$\sigma_r(\mathcal{R}) := \left\{ \lambda \in \sigma(\mathcal{R}) \subseteq \mathbb{C} : \ker(\mathcal{R} - \lambda I) = \{0\} \textit{ and } \overline{\operatorname{Range}(\mathcal{R} - \lambda I)} \neq F^2 \right\}.$$

The properties of $\sigma(\mathcal{R})$

Main Theorem

$$\sigma(\mathcal{R}) = \sigma_p(\mathcal{R}) = \{2k + n : k \in \mathbb{N}_0\}.$$

$(\because)$ (Step 1) $\mathcal{R} : Dom(\mathcal{R}) \to F^2$ is injective.
Suppose that $\mathcal{R}f_1 = \mathcal{R}f_2 \quad \forall f_1, f_2 \in Dom(\mathcal{R})$.
Since $\{e_\alpha : \alpha \in \mathbb{N}_0^n\}$ is an orthonormal basis for F^2,

$$\sum_{\alpha \in \mathbb{N}_0^n} (2|\alpha| + n)\langle f_1, e_\alpha\rangle e_\alpha = \mathcal{R}f_1 = \mathcal{R}f_2 = \sum_{\alpha \in \mathbb{N}_0^n} (2|\alpha| + n)\langle f_2, e_\alpha\rangle e_\alpha$$

$$\Rightarrow \sum_{\alpha \in \mathbb{N}_0^n} (2|\alpha| + n)\langle f_1, e_\alpha\rangle e_\alpha = \sum_{\alpha \in \mathbb{N}_0^n} (2|\alpha| + n)\langle f_2, e_\alpha\rangle e_\alpha$$

$$\Rightarrow \sum_{\alpha \in \mathbb{N}_0^n} (2|\alpha| + n)\langle f_1 - f_2, e_\alpha\rangle e_\alpha = 0$$

$$\therefore f_1 = f_2 .$$

The proof of main theorem

Formal calculuation

(Step 2) Suppose that for $f \in Dom(\mathcal{R})$, $\mathcal{R}f = g \in F^2$.
Since $\mathcal{R}$: *injective*, $f = \mathcal{R}^{-1}g$. Then

$$g = \sum_{\alpha \in \mathbb{N}_0^n} \langle g, e_\alpha \rangle e_\alpha$$

$$\mathcal{R}f = \sum_{\alpha \in \mathbb{N}_0^n} (2|\alpha| + n) \langle f, e_\alpha \rangle e_\alpha$$

Hence

$$\langle g, e_\alpha \rangle = (2|\alpha| + n) \langle f, e_\alpha \rangle$$

$$\therefore \quad \langle f, e_\alpha \rangle = \frac{1}{(2|\alpha| + n)} \langle g.e_\alpha \rangle$$

The proof of main theorem

Formal calculuation for the proof

$$\mathcal{R}^{-1}g = f = \sum_{\alpha\in\mathbb{N}_0^n} \langle f, e_\alpha \rangle e_\alpha = \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{(2|\alpha|+n)} \langle g.e_\alpha \rangle e_\alpha$$

for any $g \in F^2$ converges in F^2.

Let $\mathcal{E}_\alpha = \frac{1}{(2|\alpha|+n)}$. Then

$$\mathcal{R}f = \mathcal{R}\left(\sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\mathcal{E}_\alpha} \langle g, e_\alpha \rangle e_\alpha \right) = \sum_{\beta\in\mathbb{N}_0^n} \mathcal{E}_\beta \left\langle \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\mathcal{E}_\alpha} \langle g, e_\alpha \rangle e_\alpha, e_\beta \right\rangle e_\beta$$

$$= \sum_{\alpha\in\mathbb{N}_0^n} \mathcal{E}_\alpha \frac{1}{\mathcal{E}_\alpha} \langle g, e_\alpha \rangle e_\alpha$$

$$= \sum_{\alpha\in\mathbb{N}_0^n} \langle g, e_\alpha \rangle e_\alpha = g$$

(Step 3) $\mathcal{R}$ is self-adjoint and
$\langle f, \mathcal{R}f\rangle \geq 0 \;\; \forall\, f \in Dom(\mathcal{R}) \Rightarrow \sigma(\mathcal{R}) \subseteq [0, \infty]$
consider $\lambda(\neq \mathcal{E}_\alpha) \in \mathbb{R}, \; \forall\, |\alpha| = \; 0, \; 1, \; 2, \cdots$ and $(\lambda I - \mathcal{R})^{-1}$

$$
\begin{aligned}
&(\lambda I - \mathcal{R})^{-1} g = f \\
&\Leftrightarrow g = \lambda f - \mathcal{R} f \\
&\Leftrightarrow g = \sum_{\alpha \in \mathbb{N}_0^n} \langle g, e_\alpha \rangle e_\alpha \\
&\qquad = \sum_{\alpha \in \mathbb{N}_0^n} \left(\langle \lambda f, e_\alpha \rangle e_\alpha - \mathcal{E}_\alpha \langle f, e_\alpha \rangle e_\alpha \right) \\
&\qquad = \sum_{\alpha \in \mathbb{N}_0^n} \langle \lambda f - \mathcal{E}_\alpha f, e_\alpha \rangle e_\alpha
\end{aligned}
$$

The proof of main theorem

Repeat formal calculuation for the proof

$$\langle g, e_\alpha \rangle = \langle \lambda f - \mathcal{E}_\alpha f, e_\alpha \rangle = \lambda - \mathcal{E}_\alpha \langle f, e_\alpha \rangle$$
$$\therefore \langle f, e_\alpha \rangle = \frac{1}{\lambda - \mathcal{E}_\alpha} \langle g, e_\alpha \rangle$$

$$f = \frac{1}{\lambda}\mathcal{R}f + \frac{1}{\lambda}g = \frac{1}{\lambda}\sum_{\alpha\in\mathbb{N}_0^n} \mathcal{E}_\alpha \langle f, e_\alpha \rangle e_\alpha + \frac{1}{\lambda}\sum_{\alpha\in\mathbb{N}_0^n} \langle g, e_\alpha \rangle e_\alpha$$
$$= \frac{1}{\lambda}\sum_{\alpha\in\mathbb{N}_0^n} \frac{\mathcal{E}_\alpha}{\lambda - \mathcal{E}_\alpha} \langle g, e_\alpha \rangle e_\alpha + \frac{1}{\lambda}\sum_{\alpha\in\mathbb{N}_0^n} \langle g, e_\alpha \rangle e_\alpha$$

converges in F^2 for any $g \in F^2$.

The proof of main theorem

Repeat formal calculuation for the proof

$(\because)$ Since $\mathcal{E}_\alpha \to \infty$ *as* $|\alpha| \to \infty$, $|\frac{\mathcal{E}_\alpha}{\lambda - \mathcal{E}_\alpha}|$: bounded

Let

$$\mu := \sup_{|\alpha| \geq 0} \left| \frac{\mathcal{E}_\alpha}{\lambda - \mathcal{E}_\alpha} \right| < \infty$$

and

$$\phi_K := \sum_{|\alpha|=0}^{K} \frac{\mathcal{E}\alpha}{\lambda - \mathcal{E}_\alpha} \langle g, e_\alpha \rangle e_\alpha := \sum_{\kappa=0}^{K} \sum_{|\alpha|=\kappa} \frac{\mathcal{E}\alpha}{\lambda - \mathcal{E}_\alpha} \langle g, e_\alpha \rangle e_\alpha$$

Then for $K_2 > K_1$,

$$\|\phi_{K_2} - \phi_{K_1}\|^2 = \left\langle \sum_{|\alpha|=K_1+1}^{K_2} \frac{\mathcal{E}_\alpha}{\lambda - \mathcal{E}_\alpha} \langle g, e_\alpha \rangle e_\alpha, \sum_{|\beta|=K_1+1}^{K_2} \frac{\mathcal{E}_\beta}{\lambda - \mathcal{E}_\beta} \langle g, e_\beta \rangle e_\beta \right\rangle$$

$$= \sum_{|\alpha|=K_1+1}^{K_2} \left| \frac{\mathcal{E}_\alpha}{\lambda - \mathcal{E}_\alpha} \right|^2 |\langle g, e_\alpha \rangle|^2 \leq \mu^2 \sum_{|\alpha|=K_1+1}^{K_2} |\langle g, e_\alpha \rangle|^2 \quad \cdots \quad \text{Cauchy Seq}$$

Otherwise

$$\sum_{|\alpha|=0}^{\infty} |\langle g, e_\alpha \rangle|^2 = \|g\|_{F^2}^2,$$

Therefore $(\lambda I - \mathcal{R})^{-1}$ is defined everywhere in F^2.

The proof of main theorem

Repeat formal calculuation for the proof

(Step 4) $(\lambda I - \mathcal{R})^{-1}$ is bounded

$$\|(\lambda I - \mathcal{R})^{-1})g\| \leq \frac{1}{\lambda}\|g\| + \frac{1}{\lambda}\mu\|g\| = \frac{(1+\mu)}{\lambda}\|g\|.$$

For $\lambda(\neq \mathcal{E}_\alpha) \in \mathbb{R}$, $\exists$ $(\lambda I - \mathcal{R})$: bounded, so $\lambda \in \rho(\mathcal{R})$.

$\therefore \sigma(\mathcal{R}) = \sigma_p(\mathcal{R})$.

□

The fractional radial derivative

Now, we can define the fractional radial derivative by using the following theorem.

Functional Calculus

For any self-adjoint operator $\mathcal{R}$ and measurable function g, define a (possibly unbounded) operator, denoted by $g(\mathcal{R})$, by

$$g(\mathcal{R}) = \int_{\sigma(\mathcal{R})} g(\lambda) d\mu^{\mathcal{R}}(\lambda)$$

where $\mu^{\mathcal{R}}$ is a projection-valued measure (or spectral measure) .

Let $g(x) = x^s, \ s \in \mathbb{R}$. Then by Functional Calculus,

$$g(\mathcal{R})(f)(z) = \mathcal{R}^s f(z) = \sum_{\alpha \in \mathbb{N}_0^n} (2|\alpha| + n)^s c_\alpha e_\alpha(z).$$

Def of Semigroup and so on

To show that a relationship between $\mathcal{R}$ and a semigroup , we need a some definition.

Semigroup

Let X be Banach space and $\{B_t\}_{t\geq 0}$ be a family of linear, bounded operators
$B_t \; : \; X \; \rightarrow \; X$ for all $t \geq 0$.
$\{B_t\}_{t\geq 0}$ is called a *Semigroup* iff $B_0 = I$ and $B_{s+t} = B_s B_t \; \forall \; t, s \; \geq 0$.

Strongly continuous semigroup

A semigroup $\{B_t\}_{t\geq 0}$ is called *a strongly continuous semigroup* (or $\mathcal{C}_0 -$ *semigroup*) iff

$$\|B_t f \; - \; f\| \; \rightarrow \; 0 \quad \text{as} \quad t \; \rightarrow \; 0^+, \forall \, f \in X.$$

Def of Semigroup and so on

Contractive or Contraction Semigroup

Let $\{B_t\}_{t\geq 0}$ be a strongly continuous semigroup on X. Then $\{B_t\}_{t\geq 0}$ is called a *Contractive* or *Contraction Semigroup* iff $\|B_t\| \leq 1$, $\forall\, t \geq 0$.

Infinitesimal Generator

Let $\{B_t\}_{t\geq 0}$ be a semigroup on X. Set

$$Dom(A) := \left\{ f \in X : \exists \lim_{t\to 0^+} \frac{B_t f - f}{t} \right\}$$

$$Af := \lim_{t\to 0^+} \frac{B_t f - f}{t} \text{ for } f \in Dom(A)$$

Then A is called the infinitesimal generator of $\{B_t\}_{t\geq 0}$.

Between $\mathcal{R}$ and $\{B_t\}_{t\geq 0}$

Let $\{B_t\}_{t\geq 0}$ defined by the expansion.

$$B_t f(z) = \sum_{\alpha\in\mathbb{N}_0^n} e^{-(2|\alpha|+n)t} c_\alpha e_\alpha(z) \text{ for } f \in F^2.$$

Then $\{B_t\}_{t\geq 0}$ has a following properties.

The properties of $\{B_t\}_{t\geq 0}$

(1) B_t is a bounded operator and $\{B_t\}_{t\geq 0}$ is a strongly continuous semigroup.
(2) $\{B_t\}_{t\geq 0}$ is a contractive semigroup.
(3) $-\mathcal{R}$ is the infinitesimal generator of $\{B_t\}_{t\geq 0}$.
(4) $B_t f = e^{-t\mathcal{R}} f, \ \forall\ f \in F^2$

Proof on $\{B_t\}_{t \geq 0}$

($\because$) (1) and (2)

$$\text{For } f \in F^2,\ B_0 f = f \Rightarrow B_0 = id_{F^2} \text{ and } \forall\, t, s \geq 0,$$

$$B_t B_s f = B_t \left(\sum_{\alpha \in \mathbb{N}_0^n} e^{-(2|\alpha|+n)s} c_\alpha e_\alpha \right)$$

$$= \sum_{\beta \in \mathbb{N}_0^n} e^{-(2|\beta|+n)t} \left\langle \sum_{\alpha \in \mathbb{N}_0^n} e^{-(2|\alpha|+n)s} c_\alpha e_\alpha , e_\beta \right\rangle e_\beta$$

$$= \sum_{\alpha \in \mathbb{N}_0^n} e^{-(2|\alpha|+n)(t+s)} c_\alpha e_\alpha = \sum_{\alpha \in \mathbb{N}_0^n} e^{-(2|\alpha|+n)(s+t)} c_\alpha e_\alpha$$

$$= \sum_{\alpha \in \mathbb{N}_0^n} e^{-(2|\alpha|+n)s} \left\langle \sum_{\beta \in \mathbb{N}_0^n} e^{-(2|\beta|+n)t} c_\beta e_\beta , e_\beta \right\rangle e_\alpha$$

$$= B_s \left(\sum_{\beta \in \mathbb{N}_0^n} e^{-(2|\beta|+n)s} c_\beta e_\beta \right) = B_s B_t f \Rightarrow\ B_t B_s f = B_s B_t f$$

$\therefore$ $\{B_t\}$ is a semigroup.

Proof on $\{B_t\}_{t\geq 0}$

$$\|B_t f\|_{F^2}^2 = \left\langle \sum_{\alpha\in\mathbb{N}_0^n} e^{-(2|\alpha|+n)t} c_\alpha e_\alpha, \sum_{\beta\in\mathbb{N}_0^n} e^{-(2|\beta|+n)s} c_\beta e_\beta \right\rangle$$
$$= \sum_{\alpha\in\mathbb{N}_0^n} e^{-2(2|\alpha|+n)t} |c_\alpha|^2 \leq e^{-2nt} \left(\sum_{\alpha\in\mathbb{N}_0^n} |c_\alpha|^2 \right)$$
$$= e^{-2nt}\|f\|_{F^2} \leq \|f\|_{F^2},\ t \geq 0$$
$$\therefore\ B_t \in B(F^2, F^2) \text{ and contractive semigroup.}$$

To show that $\{B_t\}_t \geq 0$ is a str contiuous semigroup, we need a following definition.

Discrete Measure

$$\mu = \sum_{k=1}^{\infty} |c_k|^2 \delta_k,\ \delta_k \in \mathcal{N}$$

Proof on $\{B_t\}_{t\geq 0}$

$$B_t f - f = \sum_{\alpha \in \mathbb{N}_0^n} \left(e^{-(2|\alpha|+n)t} - 1 \right) c_\alpha e_\alpha$$

$$\|B_t f - f\|_{F^2}^2 = \sum_{\alpha \in \mathbb{N}_0^n} \left| e^{-(2|\alpha|+n)t} - 1 \right|^2 |c_\alpha|^2$$

Let $|\alpha| := k$,

$$\lim_{t\to 0^+} \|B_t f - f\|_{F^2}^2 = \lim_{t\to 0^+} \sum_{k=0}^{\infty} \left| e^{-(2k+n)t} - 1 \right|^2 |c_k|^2 = \lim_{t\to 0^+} \int_0^\infty \left| e^{-(2k+n)t} - 1 \right|^2 d\mu(\lambda)$$

μ : discrete measure and by lebesgue dominate convergence,

$$= \int_0^\infty \left(\lim_{t\to 0^+} \left| e^{-(2k+n)t} - 1 \right|^2 \right) d\mu(\lambda) = \int_0^\infty 0 \cdot d\mu(\lambda) = 0$$

$\therefore \{B_t\}_{t\geq 0}$ is a strongly continuous semigroup.

Proof on $\{B_t\}_{t\geq 0}$

$(\because)$ (3)

$$\left\|\frac{B_t f - f}{t} - (-\mathcal{R}f)\right\|_{F^2}^2 = \left\langle \sum_{\alpha\in\mathbb{N}_0^n}\left(\frac{e^{-2(|\alpha|+n)t}-1}{t}\right)c_\alpha e_\alpha + \sum_{\alpha\in\mathbb{N}_0^n} 2(|\alpha|+n)c_\alpha e_\alpha, \right.$$

$$\left. \sum_{\beta\in\mathbb{N}_0^n}\left(\frac{e^{-2(|\beta|+n)t}-1}{t}\right)c_\beta e_\beta + \sum_{\beta\in\mathbb{N}_0^n} 2(|\beta|+n)c_\beta e_\beta \right\rangle$$

$$= \sum_{k=0}^{\infty}\left|\frac{e^{-(2k+n)t}-1}{t} + (2k+n)\right|^2 |c_k|^2$$

$$= \int_0^\infty \left|\frac{e^{-(2k+n)t-1}}{t} + (2k+n)\right|^2 d\mu(\lambda)$$

$$= \int_0^\infty \left|\frac{e^{-(2k+n)t}-1}{(2k+n)t} + 1\right|^2 (2k+n)^2 d\mu(\lambda)$$

$$= \int_0^\infty \left|\frac{e^{-(2k+n)t}-1}{(2k+n)t} + 1\right|^2 d\nu(\lambda) \quad (\because)\ \nu := \sum_{k=0}^{\infty}(2k+n)^2|c_k|^2\delta_k$$

Proof on $\{B_t\}_{t\geq 0}$

$$
\begin{aligned}
\lim_{t\to 0^+}\left\|\frac{B_t f - f}{t} - (-\mathcal{R}f)\right\|_{F^2}^2 &= \lim_{t\to 0^+}\int_0^\infty \left|\frac{e^{-(2k+n)t}-1}{(2k+n)t}+1\right|^2 d\nu(\lambda)\\
&= \int_0^\infty \lim_{t\to 0^+}\left|\frac{e^{-(2k+n)t}-1}{(2k+n)t}+1\right|^2 d\nu(\lambda) \quad \text{by L.D.C}\\
&= \lim_{t\to 0^+} 0\cdot d\nu(\lambda) = 0.
\end{aligned}
$$

Proof on $\{B_t\}_{t\geq 0}$

$(\because)$ (4) Let $g(x) = e^{-tx}$.

Then

$$e^{-t\mathcal{R}} f = e^{-t\mathcal{R}} \left(\sum_{\alpha \in \mathbb{N}_0^n} c_\alpha e_\alpha \right) = \sum_{\alpha \in \mathbb{N}_0^n} e^{-t(2|\alpha|+n)} c_\alpha e_\alpha := B_t f,$$

by The Functional Calculus.

□

Conclusion and Further work

There are other ways to define the fractional operator (Fourier Series, Fourier Transform, Quantization Map , *etc*). But we think that in this case, this method is a simple and clear because the spectrum of the radial derivative consists only eigenvalues. We will research the fractional Fock-Sobolev space and the fractional operator defined on these space.

References

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Thanks

Thank you for your attention !!

Q & A

Any question or comment?