Improper Coloring Sparse Graphs on Surfaces

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Based on results and discussions with...
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L. Esperet
F. Dross, M. Montassier, P. Ochem

May 16, 2016

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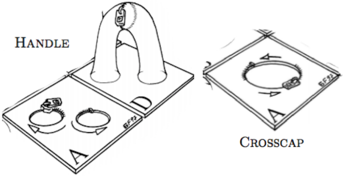

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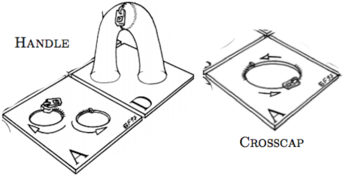

An orientable surface: add ≥ 0 handles to the sphere
A non-orientable surface: add ≥ 1 cross-caps to the sphere

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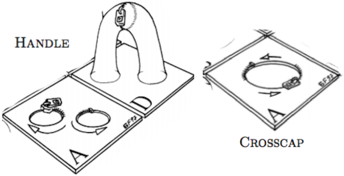

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Classification of Surfaces (Möbius 1870, von Dyck 1888, Rado 1925)

A surface is either orientable or non-orientable.

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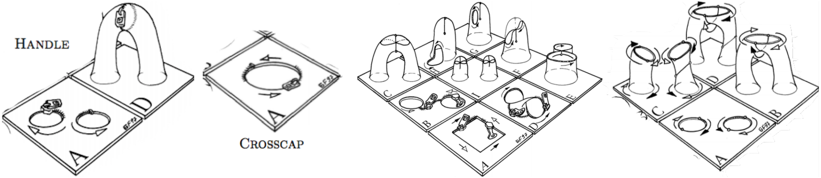

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Lemma (von Dyck 1888)

One handle and one cross-cap is equivalent to three cross-caps.

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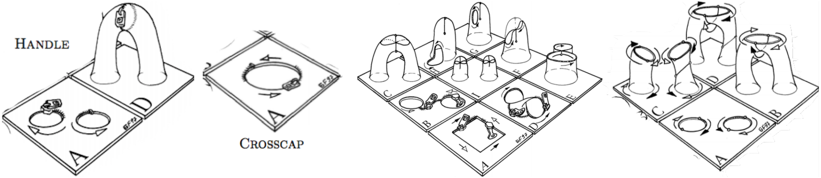

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Euler genus γ of a surface = the number of cross-caps + $2\times$handles
S_γ: a surface of Euler genus γ

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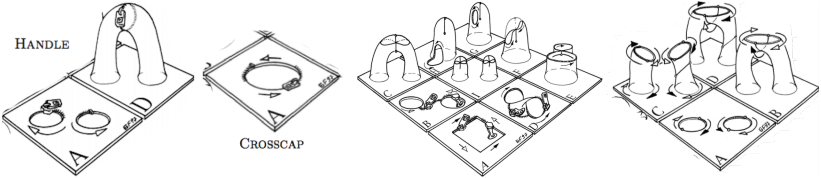

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Euler genus γ of a surface = the number of cross-caps + 2×handles
S_γ: a surface of Euler genus γ
S_0: sphere / S_1: projective plane / S_2: torus or Klein Bottle...
planar graph ⇔ graph (embeddable) on S_0 (without edges crossings)

A graph G is k-colorable if the following is possible:

- each vertex receives a color from $\{1, \ldots, k\}$
- adjacent vertices receive different colors

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OR

– partition the vertex set of G into k parts

– each part has maximum degree at most 0

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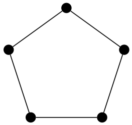

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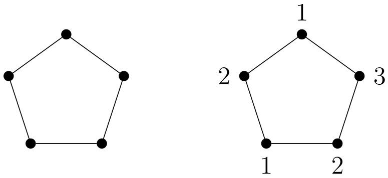

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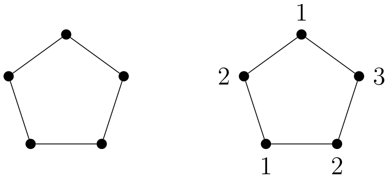

A graph G is $(d_1, \ldots, d_k)$-colorable if the following is possible:
- partition the vertex set of G into k parts
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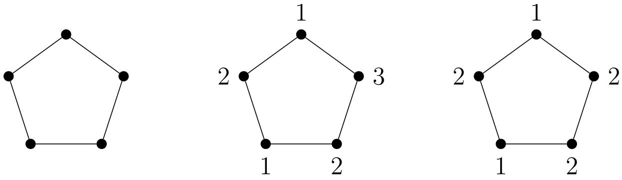

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Theorem (Appel–Haken 1977)

Every planar graph is 4-colorable.

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Every planar graph is 4-colorable ($\Leftrightarrow$ $(0, 0, 0, 0)$*-colorable*).

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Theorem (Cowen–Cowen–Woodall 1986)

Every planar graph is $(2, 2, 2)$*-colorable.*

Theorem (Eaton–Hull 1999, Škrekovski 1999)

Given k *and* ℓ*, there exists a non-*$(1, k, \ell)$*-colorable planar graph.*

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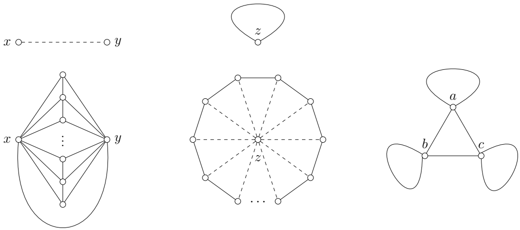

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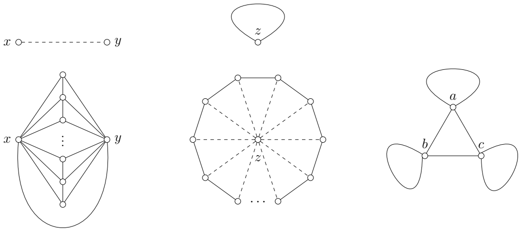

$\{x, y\}$ cannot be colored $\{k, \ell\}$ z cannot be neither k nor ℓ

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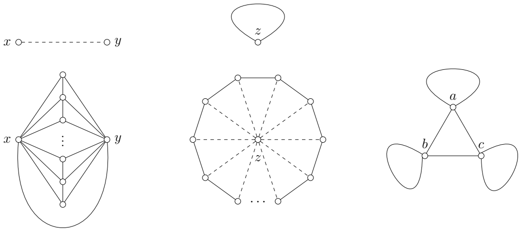

Improper coloring planar graphs with at least three parts: *SOLVED*!

Improper coloring planar graphs with two parts..........

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Cowen–Cowen–Woodall 86, Škrekovski 99 00, Montassier–Ochem 15
Given (d_1, d_2), there exists a non-(d_1, d_2)-colorable planar graph!

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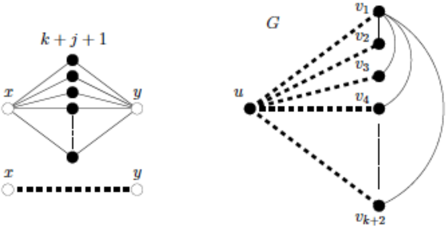

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Problem (1)

Given (d_1, d_2), determine the minimum $g = g(d_1, d_2)$ such that every planar graph with girth $\geq g$ is (d_1, d_2)-colorable.

Problem (2)

Given $(g; d_1)$, determine the minimum $d_2 = d_2(g; d_1)$ such that every planar graph with girth $\geq g$ is (d_1, d_2)-colorable.

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$\text{Mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$. If G is planar with girth g, then $\text{Mad}(G) < \frac{2g}{g-2}$.

Problem (3)

Given (d_1, d_2), determine the supremum x such that every graph with $Mad(G) \leq x$ is (d_1, d_2)-colorable.

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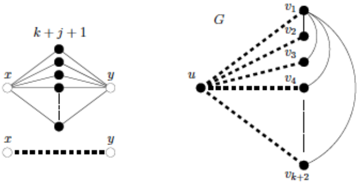
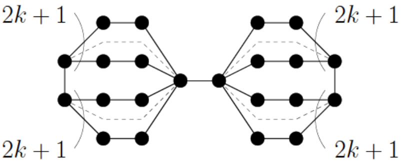


Non-(d_1, d_2)-colorable planar graph with girth 4.



Non-$(0, k)$-colorable planar graph with girth 6.

Problem (1)

Given (d_1, d_2)*, determine the minimum* $g = g(d_1, d_2)$ *such that every planar graph with girth* $\geq g$ *is* (d_1, d_2)*-colorable.*

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Given (d_1, d_2), determine the minimum $g = g(d_1, d_2)$ such that every planar graph with girth $\geq g$ is (d_1, d_2)-colorable.

$d_2 \setminus d_1$	0	1	2	3	4	5
0	$\times$					
1						
2						
3						
4						
5						
6						

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Theorem (Škrekovski 2000)

$g(d, d) = 5$ *for* $d \geq 4$

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$g(d, d) = 5$ *for* $d \geq 4$

$g(d_1, d_2) = 5$ for $\min\{d_1, d_2\} \geq 4$ since $g(d_1, d_2 + 1) \leq g(d_1, d_2)$.

Problem (1)

Given (d_1, d_2)*, determine the minimum* $g = g(d_1, d_2)$ *such that every* ***planar*** *graph with girth* $\geq g$ *is* (d_1, d_2)*-colorable.*

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0	×					
1						
2						
3						
4					5	
5					5	5
6			5	5	5	5

Theorem (Škrekovski 2000, Borodin–Kostochka 2011)

$g(d, d) = 5$ *for* $d \geq 4$ *and* $g(2, 6) = 5$

$g(d_1, d_2) = 5$ for $\min\{d_1, d_2\} \geq 4$ since $g(d_1, d_2 + 1) \leq g(d_1, d_2)$.

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0	$\times$					
1						
2						
3						
4					5	
5					5	5
6			5	5	5	5

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$d_2 \setminus d_1$	0	1	2	3	4	5
0	$\times$					
1						
2	8					
3						
4	7				5	
5	7				5	5
6	7		5	5	5	5

Theorem (Montassier–Ochem 2015, Borodin–Kostochka 2011, 2014)

$g(0, k) = 7$ *for* $k \geq 4$
$g(0, 2) = 8$

Problem (1)

Given (d_1, d_2), determine the minimum $g = g(d_1, d_2)$ such that every planar graph with girth $\geq g$ is (d_1, d_2)-colorable.

$d_2 \setminus d_1$	0	1	2	3	4	5
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Problem (1)

Given (d_1, d_2), *determine the minimum* $g = g(d_1, d_2)$ *such that every* *planar* *graph with* *girth* $\geq g$ *is* (d_1, d_2)*-colorable.*

$d_2 \setminus d_1$	0	1	2	3	4	5
0	$\times$					
1	10 or 11					
2	8					
3						
4	7				5	
5	7				5	5
6	7		5	5	5	5

Effort to determine $g(0, 1)$.....

$g(0, 1) \leq 16$ 2007 Glebov–Zambalaeva
$g(0, 1) \leq 14$ 2009 Borodin–Ivanova
$g(0, 1) \leq 12$ 2011 Borodin–Kostochka
$g(0, 1) \geq 10$ 2013 Esperet–Montassier–Ochem–Pinlou
$g(0, 1) \leq 11$ 2014 Kim–Kostochka–Zhu

Problem (1)

Given (d_1, d_2), determine the minimum $g = g(d_1, d_2)$ such that every planar graph with girth $\geq g$ is (d_1, d_2)-colorable.

$d_2 \setminus d_1$	0	1	2	3	4	5
0	$\times$					
1	10 or 11	6 or 7				
2	8	6 or 7	5 or 6			
3	7 or 8	6 or 7	5 or 6	5 or 6		
4	7	5 or 6	5 or 6	5 or 6	5	
5	7	5 or 6	5 or 6	5 or 6	5	5
6	7	5 or 6	5	5	5	5

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4	7	5 or 6	5 or 6	5 or 6	5	
5	7	5 or 6	5 or 6	5 or 6	5	5
6	7	5 or 6	5	5	5	5

No value of $g(1, d_2)$ was determined!

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Question (Raspaud 2013)

Is a planar graph with girth ≥ 5 indeed (d_1, d_2)-colorable for all $d_1 + d_2 \geq 8$?

Problem (1)

Given (d_1, d_2), determine the minimum $g = g(d_1, d_2)$ such that every planar graph with girth $\geq g$ is (d_1, d_2)-colorable.

$d_2 \setminus d_1$	0	1	2	3	4	5
0	×					
1	10 or 11	6 or 7				
2	8	6 or 7	5 or 6			
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5	7	5 or 6	5 or 6	5 or 6	5	5
6	7	5 or 6	5	5	5	5

Question (Raspaud 2013, Montassier–Ochem 2015)

Is a planar graph with girth ≥ 5 indeed (d_1, d_2)-colorable for all $d_1 + d_2 \geq 8$?
Is there a d_2 such that $g(1, d_2) = 5$?

Problem (1)

Given (d_1, d_2), determine the minimum $g = g(d_1, d_2)$ such that every planar graph with girth $\geq g$ is (d_1, d_2)-colorable.

$d_2 \setminus d_1$	0	1	2	3	4	5
0	$\times$					
1	10 or 11	6 or 7				
2	8	6 or 7	5 or 6			
3	7 or 8	6 or 7	5 or 6	5 or 6		
4	7	5 or 6	5 or 6	5 or 6	5	
5	7	5 or 6	5 or 6	5	5	5
6	7	5 or 6	5	5	5	5

Theorem (C.–Raspaud 2015)

$g(3, 5) = 5$. *Every planar graph with girth ≥ 5 is $(3, 5)$-colorable.*

Problem (1)

Given (d_1, d_2), determine the minimum $g = g(d_1, d_2)$ such that every planar graph with girth $\geq g$ is (d_1, d_2)-colorable.

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0	×					
1	10 or 11	6 or 7				
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4	7	5 or 6	5 or 6	5 or 6	5	
5	7	5 or 6	5 or 6	5	5	5
6	7	5 or 6	5	5	5	5

Theorem (C.–Raspaud 2015)

$g(3,5) = 5$. *Every planar graph with girth ≥ 5 is $(3,5)$-colorable.*

Theorem (Choi–C.–Jeong–Suh 2016+)

$g(1,10) = 5$

Problem (1)

Given (d_1, d_2), determine the minimum $g = g(d_1, d_2)$ such that every planar graph with girth $\geq g$ is (d_1, d_2)-colorable.

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0	$\times$					
1	10 or 11	6 or 7				
2	8	6 or 7	5 or 6			
3	7 or 8	6 or 7	5 or 6	5 or 6		
4	7	5 or 6	5 or 6	5 or 6	5	
5	7	5 or 6	5 or 6	5	5	5
6	7	5 or 6	5	5	5	5

Theorem (C.–Raspaud 2015)

$g(3, 5) = 5$. *Every planar graph with girth ≥ 5 is $(3, 5)$-colorable.*

Theorem (Choi–C.–Jeong–Suh 2016+)

$g(1, 10) = 5$. *Every planar graph with girth ≥ 5 is $(1, 10)$-colorable.*

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Given (d_1, d_2), determine the minimum $g = g(d_1, d_2)$ such that every planar graph with girth $\geq g$ is (d_1, d_2)-colorable.

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3	7 or 8	6 or 7	5 or 6	5 or 6		
4	7	5 or 6	5 or 6	5 or 6	5	
5	7	5 or 6	5 or 6	5	5	5
6	7	5 or 6	5	5	5	5
⋮	⋮	⋮	⋮	⋮	⋮	⋮
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Theorem (Choi–C.–Jeong–Suh 2016+)

$g(1, 10) = 5$. *Every planar graph with girth ≥ 5 is $(1, 10)$-colorable.*

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Given (d_1, d_2), determine the minimum $g = g(d_1, d_2)$ such that every planar graph with girth $\geq g$ is (d_1, d_2)-colorable.

$d_2 \setminus d_1$	0	1	2	3	4	5
0	×					
1	10 or 11	6 or 7				
2	8	6 or 7	5 or 6			
3	7 or 8	6 or 7	5 or 6	5 or 6		
4	7	5 or 6	5 or 6	5 or 6	5	
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Only finitely many values left!

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Tightness example:

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Tightness example: Goal: construct a non-$(1, k)$-colorable graph on $S_{O(k)}$

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A triple is three vertices that induces at most one edge.
Given a triple, let "adding a P_3" mean the following:

Obtain G_k in the following way:
– Start with C_7.
– Do the operation of adding a P_3 to each triple $3k + 1$ times.

In a $(1, k)$-coloring of C_7, there must be a triple T all colored with k.
At least one P_3 that was added to T cannot have a vertex of color k.

G_k has $7 + 5(3k + 1) \cdot \left(\binom{7}{3} - 7\right)$ edges, so the Euler genus is linear in k.

Graphs on surfaces!

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S_γ: a surface of Euler genus γ

planar graph $\Leftrightarrow$ graph (embeddable) on S_0

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Theorem (Appel–Haken 1977)

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Theorem (Archdeacon 87, Cowen–Cowen–Jesurum 97, Woodall 2011)

Every graph on S_γ is (c_3, c_3, c_3)-colorable with $c_3 = \max\{15, \frac{3\gamma-8}{2}\}$.
with $c_3 = \max\{12, 6 + \sqrt{6\gamma}\}$.
with $c_3 = \max\{9, 2 + \sqrt{4\gamma + 6}\}$.

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Improper coloring graphs on surfaces: *SOLVED*!

Improper coloring sparser graphs on surfaces................

Improper coloring sparser graphs on surfaces................ girth condition!

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Improper coloring sparser graphs on surfaces............... girth condition!

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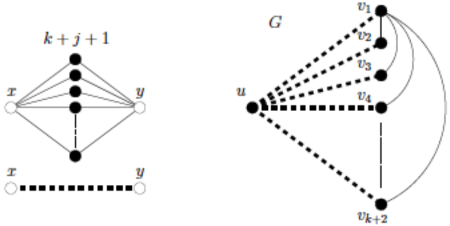

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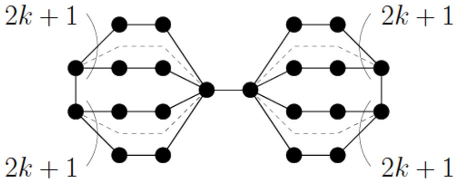

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Every graph on S_γ with girth ≥ 7 is $(0, 5 + \lceil \sqrt{14\gamma + 22} \rceil)$-colorable.

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Improper coloring graphs on surfaces with girth conditions: *SOLVED*!

Theorem (C.–Esperet 2016++, Choi–C.–Jeong–Suh 2016+)

A graph on S_γ *with* $\gamma > 0$ *is* $(0, 0, 0, 9\gamma - 4)$*-colorable.*
A graph on S_γ *with* $\gamma > 0$ *is* $(2, 2, 9\gamma - 4)$*-colorable.*
A graph on S_γ *with girth* ≥ 4 *is* $(0, 0, \lceil \frac{10\gamma+32}{3} \rceil)$*-colorable.*
A graph on S_γ *with girth* ≥ 5 *is* $(1, \max\{10, \lceil \frac{12\gamma+47}{7} \rceil\})$*-colorable.*
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There exists a non-$(1, k, \ell)$-colorable planar graph.
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Lemma (C.–Esperet 2016++)

If v is a vertex of a connected graph G on S_γ with $\gamma > 0$, then there exists a connected subgraph H containing v such that G/H is planar and every vertex of G has at most $9\gamma - 4$ neighbors in H.

Future directions.............

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3	7 or 8	6 or 7	5 or 6	5 or 6		
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Is there another “jump” besides between $g(0, 1)$ and $g(0, 2)$?!

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Theorem (Cowen–Goddard–Jesurum 1997)

Every toroidal graph is $(1,1,1,1,1)$-colorable and $(2,2,2)$-colorable.

Question: Is every toroidal graph $(1,1,1,1)$-colorable?

Thank you for your attention!