Association Schemes All Of Whose Symmetric Fusions Are Integral

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Definition. (Association Scheme)

Let X be a finite set and S a partition of $X \times X$. We say that the pair (X, S) is an **association scheme** (or shortly **scheme**) if it satisfies the following:

(i) $\{(x, x) \mid x \in X\} \in S$;

(ii) For $s \in S$, $s^* = \{(y, x) \mid (x, y) \in s\} \in S$;

(iii) For $s, t, u \in S$, the number of $\{z \mid (x, z) \in s, (z, y) \in t\}$ is constant whenever $(x, y) \in u$. The constant is denoted by a_{stu}.

We say (X, S) is **symmetric** if for each $s \in S$, $s = s^*$.
A scheme (X, T) is a **fusion** of (X, S) if for each $t \in T$, t is a union of elements of S.

Definition. (Adjacency Matrix)

Let (X, S) be a scheme. For $s \in S$, the **adjacency matrix** of s is defined by

$$(\sigma_s)_{x,y} = \begin{cases} 1, & \text{if } (x, y) \in s, \\ 0, & \text{otherwise.} \end{cases}$$

A scheme (X, S) is called **integral** if for any $s \in S$, $\mathrm{ev}(\sigma_s) \subset \mathbb{Z}$ where $\mathrm{ev}(\sigma_s)$ is the set of all eigenvalues of σ_s.

Remark.

(X, S) is symmetric iff for any $s \in S$, σ_s is symmetric.

Example 1. (Finite Groups)

Let G be a finite group. Then $(G, \tilde{G})$ is a scheme where

$$\tilde{G} = \{\tilde{g} \mid g \in G\} \qquad \text{and} \qquad \tilde{g} = \{(a, b) \in G \times G \mid ag = b\}.$$

- Since $\tilde{G}$ contains $\tilde{e}$ and $\widetilde{g^{-1}} = \tilde{g}^*$ for any $g \in G$, (i) and (ii) are clear.
- For $g_1, g_2, g_3 \in G$ and $(x, y) \in \tilde{g_3}$, we have $a_{\tilde{g_1}\tilde{g_2}\tilde{g_3}} = |\{z \mid (x, z) \in \tilde{g_1}, (z, y) \in \tilde{g_2}\}| = \delta_{g_1 g_2, g_3}$, where δ is the Kronecker delta. So (iii) is also satisfied. Thus $(G, \tilde{G})$ is a scheme.

Example 2. (The Dihedral Group of Order 8)

Let $G = D_8$ be the dihedral group of order 8. $(G, \tilde{G})$ has a symmetric fusion scheme but not integral.

Let $G = \langle a, b \rangle$ with $o(a) = 4, o(b) = 2, bab = a^{-1}$. Then $G = \{1_G, a, a^2, a^3, b, ba, ba^2, ba^3\}$ and the elements of $\tilde{G}$ are

- $s_0 := \tilde{1_G} = \{(g, g) \mid g \in G\}$,
- $s_1 := \tilde{a} = \{(1_G, a), (a, a^2), (a^2, a^3), (a^3, 1_G),$
  $(b, ba), (ba, ba^2), (ba^2, ba^3), (ba^3, b)\}$,
- $s_2 := \tilde{a^2} = \{(1_G, a^2), (a, a^3), (a^2, 1_G), (a^3, a),$
  $(b, ba^2), (ba, ba^3), (ba^2, b), (ba^3, ba)\}$,

  $\vdots$

So we have $\tilde{G} = \{s_0, s_1, \ldots, s_7\}$.

Consider the matrix $0\sigma_{s_0} + 1\sigma_{s_1} + \cdots + 7\sigma_{s_7} =$

$$\sum_{i=0}^{7} i\sigma_{s_i} = \begin{array}{c} \\ 1_G \\ a \\ a^2 \\ a^3 \\ b \\ ba \\ ba^2 \\ ba^3 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1_G & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \end{array} \\ \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 0 & 1 & 2 & 5 & 6 & 7 & 4 \\ 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ 1 & 2 & 3 & 0 & 7 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ 5 & 6 & 7 & 4 & 3 & 0 & 1 & 2 \\ 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ 7 & 4 & 5 & 6 & 1 & 2 & 3 & 0 \end{pmatrix} \end{array}.$$

Consider the matrix $0\sigma_{s_0} + 1\sigma_{s_1} + \cdots + 7\sigma_{s_7} =$

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Note that $\sigma_{s_1}, \sigma_{s_3}$ are not $\boxed{\text{symmetric}}$.

If we take $t_0 := s_0, \quad t_1 := s_1 \cup s_3, \quad t_2 := s_2, \quad t_3 := s_4 \cup s_5, \quad t_4 := s_6 \cup s_7,$ then $(G, \{t_0, t_1, t_2, t_3, t_4\})$ is a symmetric fusion scheme of $(G, \tilde{G})$.

$$\sum_{i=0}^{4} i\sigma_{t_i} = \begin{pmatrix} 0 & 1 & 2 & 1 & 3 & 3 & 4 & 4 \\ 1 & 0 & 1 & 2 & 3 & 4 & 4 & 3 \\ 2 & 1 & 0 & 1 & 4 & 4 & 3 & 3 \\ 1 & 2 & 1 & 0 & 4 & 3 & 3 & 4 \\ 3 & 3 & 4 & 4 & 0 & 1 & 2 & 1 \\ 3 & 4 & 4 & 3 & 1 & 0 & 1 & 2 \\ 4 & 4 & 3 & 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 3 & 4 & 1 & 2 & 1 & 0 \end{pmatrix}$$

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$$\sum_{i=0}^{4} i\sigma_{t_i} = \begin{pmatrix} 0 & 1 & 2 & 1 & 3 & 3 & 4 & 4 \\ 1 & 0 & 1 & 2 & 3 & 4 & 4 & 3 \\ 2 & 1 & 0 & 1 & 4 & 4 & 3 & 3 \\ 1 & 2 & 1 & 0 & 4 & 3 & 3 & 4 \\ 3 & 3 & 4 & 4 & 0 & 1 & 2 & 1 \\ 3 & 4 & 4 & 3 & 1 & 0 & 1 & 2 \\ 4 & 4 & 3 & 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 3 & 4 & 1 & 2 & 1 & 0 \end{pmatrix}$$

But not integral since $\mathrm{ev}(\sigma_{t_3}) = \{0, 0, \pm 2, \pm\sqrt{2}, \pm\sqrt{2}\}$.

Main Problem

Definition.

Let G be a finite group. We say that G is **desired** if any symmetric fusion scheme of $(G, \tilde{G})$ is integral. If G is not desired then it is called **undesired**.

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Problem.

Classify all desired groups.

Cayley Integral Groups

A graph is called **integral** if its adjacency matrix has only integral eigenvalues. Let G be a finite group and a subset H in $G \setminus \{1_G\}$ such that if $h \in H$, then $h^{-1} \in H$. The **undirected Cayley graph** $\mathrm{Cay}(G, H)$ of G over the set H is the graph whose vertex set is G and two vertices a and b are adjacent whenever $ah = b$ for some $h \in H$. We call a finite group G **Cayley integral** whenever all undirected Cayley graphs over G are integral.
In 2010[1]- 2014[2], all finite Cayley integral groups are completely classified by many authors.

[1] Klotz and Sander, “Integral Cayley graphs over abelian groups”.
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Remark.

Every finite Cayley integral group is desired.

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Cayley Integral Groups

Theorem. A. Abdollahi M. Jazaeri (2014)

A finite group is Cayley integral if and only if it is isomorphic to one of the following:

(i) an abelian group whose exponent dividing 4 or 6;

(ii) $Q_8 \times C_2^m$ for some nonnegative integer m where Q_8 is the quaternion group and C_n is the cyclic group of degree n;

(iii) S_3 where S_n is the symmetric group of degree n;

(iv) $C_3 \rtimes C_4 = \langle x, y \mid x^3 = y^4 = 1, y^{-1}xy = x^{-1} \rangle$.

Main Result

Theorem. M. Hirasaka K. Kim O (2016+)

A group is desired if and only if it is isomorphic to one of the following:

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(v) $\boxed{S_3 \times C_2}$.

Fundamental Lemmas

Lemma 1.

Any subgroup or any homomorphic image of a desired group is desired.

Lemma 2.

The order of any elements of a desired group is one of $\{1, 2, 3, 4, 6\}$. In particular, the order of a desired group is written as $2^a 3^b$ for some nonnegative integers a, b.

Circulant Matrix

Definition. (Circulant Matrix)

An $n \times n$ circulant matrix is of the form

$$\begin{pmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{pmatrix}.$$

Eigenvalues of A Real Circulant Matrix:

$$\lambda_k = c_0 + c_{n-1}\omega_k + c_{n-2}\omega_k^2 + \ldots + c_1\omega_k^{n-1}, \qquad k = 0, 1, \ldots, n-1$$

where $\omega_k = \exp\left(\frac{2\pi k\sqrt{-1}}{n}\right)$ are the n-th roots of unity.

Proof of Lemma 2

Proof. Let G be a desired group and $x \in G$ with order n. By Lemma 1, $H := \langle x \rangle$ is desired. Take $T := \{\tilde{y} \cup \widetilde{y^{-1}} \mid y \in H\}$. Then (H, T) is a symmetric fusion scheme of $(H, \tilde{H})$.
For any $\tilde{y} \cup \widetilde{y^{-1}} \in T$, $\sigma_{\tilde{y} \cup \widetilde{y^{-1}}}$ is a circulant matrix. For example,

$$\sigma_{\tilde{x} \cup \widetilde{x^{-1}}} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & & \vdots \\ 0 & 0 & 1 & 0 & \ddots & \\ \vdots & \vdots & & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & & 1 & 0 \end{pmatrix}.$$

By the well-known eigenvalues of circulant matrices, we have $\mathrm{ev}(\sigma_{\tilde{y} \cup \widetilde{y^{-1}}}) = \{2\cos(\frac{2\pi k}{n}) \mid k = 0, 1, \ldots, n-1\}$. So $n \in \{1, 2, 3, 4, 6\}$.

If G has an element with a prime order $p \notin \{1, 2, 3, 4, 6\}$, then we can find a subgroup of G such that a symmetric fusion scheme of it is not integral. This implies that G is an undesired group. ■

Undesired Groups with Small Order

Example 3. (Undesired Groups)

The following groups are undesired:

- D_8,
- A_4,
- $(C_3 \times C_3) \rtimes C_2$ by the action of the inverse map,
- $S_3 \times C_3$,
- $C_2 \times C_2 \times S_3$,
- $(C_3 \rtimes C_4) \times C_2$,
- Non-abelian groups of order 27.

Sketch of Proof of Main Theorem

Let G be a desired group.

Case 1. G is abelian.

By Lemma 2, the order of any element of G divides 4 or 6.
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Lemma.

If G is a desired non-abelian 2-group, then G is isomorphic to $Q_8 \times C_2^m$ for some nonnegative integer m.

Lemma.

If G is a desired non-abelian group such that all two involutions commute for each other, G is isomorphic to $C_3 \rtimes C_4$ unless G is a 2-group.

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Then G is isomorphic to $Q_8 \times C_2^m$ or $C_3 \rtimes C_4$.

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Let G be a desired group. If G has two non-commuting involutions, then $|G| = 6$ or 12.

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All desired non-abelian groups of order 6 or 12 are known to be $S_3, S_3 \times C_2$ or $C_3 \rtimes C_4$.

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Thus this proof is done. ■

END

Thank you ❦